Chains and Gradings

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At a certain point in (co)homology theory we will want to start conflating chains, collections of groups and their graded groups/rings. I want to make this precise.

We will start by considering the case of Abelian groups, the category of which we will simply denote by Ab.

1 What is a chain

The first step is making precise what a chain is. For our purposes we will consider only chains of Abelian groups.

Intuitively a chain of Abelian groups is just a sequence of Abelian groups, with maps between them

$$\cdots \to C_n \xrightarrow{\partial} C_{n-1} \to \cdots$$

such that $\partial^2 = 0$. This is a *diagram* of a certain shape in the category Ab with some *conditions* on the morphism between the verticies. The (one) precise definition of a "diagram" is a functor from a category with that "shape" into Ab.

The relevant shape for normal chains is what we will denote $\mathbb{Z}_\leq$, that is the category that looks like

$$\cdots \to \bullet \to \bullet \to \cdots$$

or precisely it is the category associated to the posest $(\mathbb{Z}, \leq)$. Objects are integers and there is a formal morphism between two integers n and m iff $n \leq m$. A diagram in Ab is therefore a functor $\mathbb{Z}_\leq \to \text{Ab}$. It might be sensible then to form the category of such diagrams, which we shall denote $\text{Fun}(\mathbb{Z}_\leq, \text{Ab})$, note that this category already has a notion of morphisms that are given by natural transformations.

We have a notion of morphisms between chains, they are maps on each of the groups that commute with the boundary homomorphisms. Do these two notions of morphism agree? It is clear that the commuting square of the natural transformation definition corresponds exactly to the commuting of the boundary homomorphism in this case.

Therefore we *define* chains to be a subcategory of $\text{Fun}(\mathbb{Z}_\leq, \text{Ab})$ where boundaries square to 0. Note that this is a full subcategory, but it is a strict subcategory, as there are clearly diagrams in Ab with morphisms that dont square to zero, for instance

$$0 \to \mathbb{Z} \xrightarrow{\times 1} \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{\times 3} \mathbb{Z} \to \cdots$$

which we do not call chains. We summarise:

$$\mathrm{Ch}(\mathrm{Ab}) \hookrightarrow \mathrm{Fun}(\mathbb{Z}_{\leq}, \mathrm{Ab})$$

2 What is a graded Abelian group

Again the idea of a graded Abelian group is clear, infact even definable without categorical language. A $\mathbb{Z}$ graded Abelian group G is an Abelian group that has a decomposition

$$G = \oplus_{i \in \mathbb{Z}} G_i$$

where G_i is an Abelian group (actually the fact that G is Abelian should imply that each G_i is Abelian). We can talk about the category of graded Abelian groups by specifying morphisms to be group homomorphisms that *respect the grading*, that is

$$f : G \to H \text{ such that } f(G_i) \subseteq H_i.$$

I believe that again f will necissarily be a group homomorphism on each G_i. We will denote this category by $\mathrm{Gr}_{\mathbb{Z}}\mathrm{Ab}$. check that

Note that here we really do ask for the group to be *equal* to a direct sum as classically defined for two groups. If it was just isomorphic abstractly to something with such a grading it would be less clear what the morphisms should be.

The fact that the morphisms have to send the graded pieces to each other suggests that the peices are essentially seperate in a graded Abelian group. This gives us the picture of a graded group

$$\cdots \quad G_n \quad G_{n-1} \quad G_{n-2} \quad \cdots$$

or in other words as a diagram

$$\cdots \quad \bullet \quad \bullet \quad \bullet \quad \cdots.$$

We will denote this diagram $\mathbb{Z}$. It is the category given by the set $\mathbb{Z}$, objects are integers and the only morphisms are identity morphisms. Whether or not this is intuitively clear to you or not we can make it precise.

Lemma. *There is an equivilence of categories*

$$\mathrm{Gr}_{\mathbb{Z}}\mathrm{Ab} \cong \mathrm{Fun}(\mathbb{Z}, \mathrm{Ab})$$

Proof. It is probably easist to think about the map

$$\mathrm{Fun}(\mathbb{Z}, \mathrm{Ab}) \to \mathrm{Gr}_{\mathbb{Z}}\mathrm{Ab}$$

which assigns to a diagram the direct sum of all the peices. Then you convince yourself that this is full, faithful and essentially surjective. This uses the fact that $\oplus$ is a product (maps into a graded thing are the same as maps to each of the peices).

3 The relationship between $\mathbb{Z}$'s

A forgetful functor is one that forgets something. This is not a precise definition, nor should any be given. There is a so called forgetful functor

$$\mathrm{forget} : \mathrm{Poset} \to \mathrm{Set}$$

which just forgets the ordering, this functor is essentially surjective and faithful but not *full*. It happens that this functor (only) has a *left* adjoint (it cannot have a right adjoint becuase it is not left exact (does not preserve colimits)). The left adjoint is given by sending a set S to the poset $(S, \leq)$ where $n \leq m$ iff $n = m$.

$$\text{free} \dashv \text{forget}$$

Note that here we are considering Set and Poset as large subcategories of the category of categories. That is a poset is a type of category (arrows obey the poset axioms) and a set is a type of category (only identity morphisms). This adjunction therefore induces functors between our $\mathbb{Z}$ categories

$$\text{forget} : \mathbb{Z}_{\leq} \leftrightarrow \mathbb{Z} : \text{free},$$

which we claim is again an adjunction.

Now we can apply a type of Yoneda embedding to get a diagram in Cat

$$\begin{array}{ccc} \mathbb{Z}_{\leq} & \hookrightarrow & \text{Fun}(\mathbb{Z}_{\leq}, \text{Ab}) \\ \uparrow\downarrow & & \uparrow\downarrow \\ \mathbb{Z} & \hookrightarrow & \text{Fun}(\mathbb{Z}, \text{Ab}) \end{array}$$

Note that there are tecnicalities here becuase we are hom'ing into Ab, not Set, we cannot apply the Yoneda lemma straight up. Luckily in this case we can just check that these maps are fully faithful "by eye". We claim that this is also an adjunction.

4 The relation of chains and gradings

So far we have produced a diagram in the category of categories that looks like this

$$\begin{array}{ccccc} \mathbb{Z}_{\leq} & \hookrightarrow & \text{Fun}(\mathbb{Z}_{\leq}, \text{Ab}) & \hookleftarrow & \text{Ch}(\text{Ab}) \\ F \uparrow\downarrow f & & Y(F) \uparrow\downarrow Y(f) & & \\ \mathbb{Z} & \hookrightarrow & \text{Fun}(\mathbb{Z}, \text{Ab}) & \xleftrightarrow{\sim} & \text{Gr}_{\mathbb{Z}}\text{Ab} \end{array}$$

Because the bottom arrow is an equivilence and the the composition $\text{Gr}_{\mathbb{Z}}\text{Ab} \to \text{Fun}(\mathbb{Z}, \text{Ab})\text{Fun}(\mathbb{Z}_{\leq}, \text{Ab})$ is in the image of the map $\text{Ch}(\text{Ab}) \to \text{Fun}(\mathbb{Z}_{\leq}, \text{Ab})$ we can "run around the square" in both directions to induce maps

$$\begin{array}{ccccc} \mathbb{Z}_{\leq} & \hookrightarrow & \text{Fun}(\mathbb{Z}_{\leq}, \text{Ab}) & \hookleftarrow & \text{Ch}(\text{Ab}) \\ F \uparrow\downarrow f & & Y(F) \uparrow\downarrow Y(f) & & \uparrow\downarrow \\ \mathbb{Z} & \hookrightarrow & \text{Fun}(\mathbb{Z}, \text{Ab}) & \xleftrightarrow{\sim} & \text{Gr}_{\mathbb{Z}}\text{Ab} \end{array}$$

To be more precise we have the following adjunction

$$\text{forget} : \text{Ch}(\text{Ab}) \leftrightarrow \text{Gr}_{\mathbb{Z}}\text{Ab} : \text{free}$$

sending

$$\left(\cdots \to G_i \xrightarrow{\partial} G_{i-1} \to \cdots \right) \mapsto \oplus_i G_i$$

and

$$\oplus_i G_i \mapsto \left(\cdots \to G_i \xrightarrow{0} G_{i-1} \to \cdots \right)$$

The Point: Chains and grading have an adjunction between them that in a precise sense "comes from" forgetting the ordering on $\mathbb{Z}$.

References